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THE STOCHASTIC RADIATIVE TRANSFER EQUATION: QUANTUM DAMPING, KIRCHOFFS LAW AND NLTE[1]

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Abstract:

A method is presented based on the theory of quantum damping, for deriving a self consistent but approximate form of the quantum transport for photons interacting with fully ionized electron plasma. Specifically, we propose in this paper a technique of approximately including the effects of background plasma on a photon distribution function without directly solving any kinetic equations for the plasma itself. The result is a quantum Langevin equation for the photon number operator; the quantum radiative transfer equation. A dissipation term appears which is the imaginary part of the dielectric function for an electron gas with photon mediated electron-electron interactions due to absorption and re-emission. It depends only on the initial state of the plasma. A quantum noise operator also appears as a result of spontaneous emission of photons from the electron plasma. The thermal expectation value of this noise operator yields the emissivity which is exactly of the form of the Kirchoff-Planck relation. This non-zero thermal expectation value is a direct consequence of a fluctuation-dissipation relation (FDR).

Keywords: Radiation transport, Non-local thermodynamic equilibrium, Stochastic systems

1. Introduction

In radiative plasmas, the usual governing equation for radiation transport is a semi-classical Boltzmann equation for the specific intensity with sink and

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source terms coming from emission, absorption, and scattering processes. These processes arise from the quantum mechanical interaction of matter and radiation. The standard equation is derived via a heuristic approach where quantum mechanical properties such as photon interference effects are ignored. We call this the "top-down" approach. In the absence of polarization, dispersion effects, and multi-photon effects, the standard radiative transfer equation is,

$$\frac{1}{c}\frac{\partial I_\nu(x,\Omega,t)}{\partial t} + (\Omega \bullet \nabla) I_\nu(x,\Omega,t) = j_\nu - \sigma_\nu I_\nu(x,\Omega,t)$$

Here, $I_\nu(x,\Omega,t)$ is the specific intensity defined such that $I_\nu(x,\Omega,t)/c$ is the radiation energy density per unit solid angle per unit frequency. It is proportional to the photon distribution function. The quantity j_ν is the emissivity and σ_ν is the absorptivity. In general the emissivity and absorptivity are determined by the properties of the atom-radiation interactions and depend on detailed knowledge of the atomic populations and ionization states. As is well known, invoking the assumption of LTE in which the plasma exhibits kinetic, excitation and ionization equilibrium greatly simplifies the problem. For example, a consequence of LTE is Kirchoff's Law; $j_\nu = \sigma_\nu B_\nu(T)$.

A different approach is to derive the radiation transport equation from micro-physical first principles. This is the "bottom-up" approach. Authors, such as Cannon [1], Gelinas and Ott [2], Degl'Innocenti [3], and Graziani [4] begin with the Hamiltonian of quantum electrodynamics (QED). The advantage of this method is that one begins with a formulation containing all degrees of freedom (bound and free electrons, photons, ions, interactions) with a minimum of assumptions. Absorption, emission, and scattering mechanisms arise from the fundamental QED interactions. An equation governing the time evolution of the photon number operator arises from this method. The photon number operator can be cast into an object similar to the specific intensity encountered in the heuristic approach. It is a specific intensity operator formed by using the Klimontovich operator [5]. Extracting a radiation transport equation, similar to the heuristic one discussed above, from the many body formalism, yields insights into the assumptions underlying the heuristic approach. It also provides a tool to incorporate multi-photon processes, variable refractive index, magnetic fields, etc. Since this method uses a micro-physics approach, it begins with the plasma in NLTE. Additional assumptions have to be placed on the formalism to enforce LTE.

Yet a third way, which we will call the "Middle-Road", has been advocated as a compromise between the heuristic and micro-physics approaches [6,7,4]. The RADIOM model of Busquet [6] is an example as is the work of More, Kato, Libby and Faussurier [7]. These works attempt to alter the emissivity and opacity in the heuristic transport equation so that NLTE effects can be approximately included. A different approach is to use a rigorous stochastic approach. The difficulty in dealing deterministically with the large number of microscopic elements encountered in an NLTE plasma implies that it might be useful to project all dynamics onto a "small" number of macro-variables [8,4]. The problem, which consists of electrons and photons and their interaction, is decomposed into two types of variables called "resolved" and "unresolved". The resolved or macro-variable degrees of freedom are associated with the photons. The unresolved or micro-variable degrees of freedom are associated with the electron and ions. From non-equilibrium statistical mechanics, this process implies that the resulting dynamics of the macro-variables or photons will be described by a Langevin equation [8,9]. This Langevin equation will have time-history effects. It will involve transfer of information from the macro-to-micro variables (absorption=dissipation) and it will involve transfer of information from micro-to-macro variables (emissivity=fluctuation). Finally, for a closed system (which we assume), their exists a relationship between the dissipation (absorption) and fluctuation (emissivity) through a fluctuation-dissipation relation (FDR). As we will see this takes the form of Kirchoff's Law for LTE [4].

In this paper, we present a method for deriving a self consistent but approximate form of the quantum transport for photons interacting with a fully ionized electron plasma. Specifically, what we propose in this paper is a technique of approximately including the effects of a background plasma on a photon distribution function without directly solving any kinetic equations for the plasma itself. This method is based on the Langevin description of quantum damping [10]. Like Cannon [1], Gelinas and Ott [2], and Almeida [11], we consider a closed system consisting of photons interacting with an electron gas or plasma which is itself coupled to a background of positive ions. However, unlike these authors, we separate the electron and photon degrees of freedom into two sets. The photons are treated as the system of interest. The electrons are treated as background or reservoir degrees of freedom. This means that we are willing to describe the electron plasma in some as yet unspecified statistical fashion and give up on detailed knowledge of its state. As we will find, this description implies that the electron plasma fluctuates about its equilibrium or LTE configuration due to interactions with the photons. Depending on the relaxation time scale of the electron plasma, these fluctuations in general can be large and non-Markovian. In this paper, we focus on the Markov limit. From the photon viewpoint, what results is a quantum mechanical Langevin equation for the radiative transfer equation. In this paper, the coupling of the photons to the electron plasma is performed to lowest non-trivial order in e/mc. Mathematically, the procedure is to construct the reduced Heisenberg equations of motion for the photon number operator by explicitly integrating out the electron degrees of freedom. What results is a quantum Langevin equation for the photon number operator. A dissipation term appears which is the imaginary part of the dielectric function for an electron gas with photon mediated electron-electron interactions due to scattering. It depends only on the initial state of the plasma. A quantum noise operator also appears as a result of spontaneous emission of photons from the electron plasma. The thermal expectation value of this noise operator yields the emissivity which is exactly of the form of the Kirchoff-Planck relation. This non-zero thermal expectation value is a direct consequence of a fluctuation-dissipation relation (FDR). The fluctuations of the quantum noise operator yield the NLTE deviations from the Kirchoff-Planck relation. Therefore, the electron plasma that the photons interact with acts like a damping mechanism for the radiation. The plasma provides a reservoir for the loss of photons of a given momentum while at the same time providing a source for a Planck distribution of photons. The particular model considered in this paper assumes the plasma is fully ionized. Hence, NLTE for us simply means the electrons may deviate from a Fermi-Dirac (or Maxwell-Boltzmann at high temperature). Of possibly greater interest, are partially ionized plasmas where bound states exist. The framework presented here can be extended to the case of bound electrons. For the sake of simplicity and to emphasize the method we have chosen to simply consider a fully ionized plasma..

2. Quantum Theory of Damping: An Introduction

2.1. *Quantum Damping and the Wigner-Weiskopff Approximation*

The quantum theory of damping has had along and interesting history especially in the field of quantum optics and lasers [10]. Unfortunately, it seems little known in the radiative transfer and plasma science community. Therefore, we give a simple introduction to the theory. The presentation given here is due to Liousell [12].

Consider the single mode for a radiation field interacting with a system made up of many degrees of freedom (phonons, other photon modes, etc.). Assuming a simple bi-linear coupling between the single mode radiation field and the system made up of many degrees of freedom, we have the following second quantized effective Hamiltonian written in terms of annihilation and creation operators,

$$H = \hbar\omega_C a^+ a + \hbar\sum_j \omega_j b_j^+ b_j + \hbar\sum_j \left(\kappa_j b_j a + \kappa_j^* b_j^+ a\right) \tag{1}$$

The many body degrees of freedom will be referred to as a many body reservoir. The Heisenberg equations of motion are given by,

$$\frac{dO}{dt} = \frac{i}{h}\left[H, O\right] \tag{2}$$

Assuming all degrees of freedom obey Bose-Einstein statistics, we also have the canonical commutation relations

$$\left[a(t), a^+(t)\right] = 1 \quad \left[b_j(t), b_k^+(t)\right] = \delta_{jk} \tag{3}$$

Therefore, the dynamics of the single mode radiation field and the many body reservoir are given by,

$$\frac{da(t)}{dt} = -i\omega_C a(t) - i\sum_j \kappa_j b_j(t) \qquad \frac{db_j(t)}{dt} = -i\omega_C b_j(t) - i\kappa_j^* a(t) \tag{4}$$

Our goal is to remove or projects out the unwanted or micro-variables. In this case, we assume the macro-variable or resolved degree of freedom refers to the single mode radiation field. The micro-variables or unresolved degrees of freedom refer to the many body reservoir. Due to the linearity of the equation for b_j, it may be integrated in closed form to give

$$b_j(t) = e^{-i\omega_j t} b_j(0) - i\kappa_j^* \int_0^t d\tau e^{-i\omega_j (t-\tau)} a(\tau) \qquad (5)$$

Substituting equation (4) in the equation for a (t) yields an evolution equation for the single mode radiation field that depends only on the single mode radiation field degrees of freedom and the initial state of the many body reservoir.

$$\frac{da(t)}{dt} = -i\omega_C a(t) - \sum_j \left|\kappa_j\right|^2 \int_0^t d\tau e^{-i\omega_j(t-\tau)} a(\tau) - i\sum_j \kappa_j e^{-i\omega_j t} b_j(0) \quad (6)$$

For simplicity, it is useful to transform away the oscillating component by defining $a(t) = A(t)e^{-i\omega_C t}$. We have a slightly modified equation given by,

$$\frac{dA(t)}{dt} = -\sum_j \left|\kappa_j\right|^2 \int_0^t d\tau e^{-i\omega_j(t-\tau)} A(\tau) - i\sum_j \kappa_j e^{-i\omega_j t} b_j(0) \qquad (7)$$

$$K(t-\tau) = -\sum_j \left|\kappa_j\right|^2 e^{-i\omega_j(t-\tau)} \qquad G_A(t) = -i\sum_j \kappa_j e^{-i\omega_j t} b_j(0) \quad (8)$$

We can identify two types of terms in equation (6). The first term involves an integration over the time history of A(t) multiplied by factors characteristic of the many body reservoir. The kernel of the integral is defined by K. The kernel plays the role of dissipation. It represents the loss of information from the single mode radiation field to the many body reservoir. The second term depends only on the characteristics of the many body reservoir in its initial state. It plays the role of fluctuations. It represents a source of information into the single mode radiation field from the many body reservoir. The presence of the fluctuations is extremely important. Without it, the canonical commutation relations would be violated. With it, the commutation relations are preserved for all time. There exists an exact balance of dissipation and fluctuations which preserves the commutation relations and can be thought of as a consequence of a fluctuation-dissipation relation (FDR). We will see this in section 2.2.

Equation (6) can be thought of as an equation for the projected or resolved degrees of freedom. Let us look at some of its properties

2.2. *The Markov Limit*

In general, equation (6) cannot be solved in closed form. However, if we consider the limit where the many body reservoir modes relax on a time scale much shorter than the characteristic frequency of the single mode radiation field then things simplify considerably.

Performing a Laplace transform of equation (6) yields the following closed form,

$$\tilde{A}(s) = \frac{a(0) + \tilde{G}_A(s)}{s + \sum_j \frac{\left|\kappa_j\right|^2}{s + i(\omega_j - \omega_C)}} \tag{9}$$

The inverse Laplace transform of equation (9) depends on the location of the poles or the zeros of the denominator. That is,

$$\Delta(s) = s + \sum_j \frac{\left|\kappa_j\right|^2}{s + i(\omega_j - \omega_C)} = 0 \tag{10}$$

Now, in the limit of a weak interaction between the single mode radiation field and the many body reservoir, the poles are located near s=0. Therefore, to lowest order in perturbation theory, the poles are located at,

$$\begin{aligned} \Delta(s) - s &\approx \lim_{s \to 0} \sum_j \frac{\left|\kappa_j\right|^2}{s + i(\omega_j - \omega_C)} \\ &= \sum_j \left|\kappa_j\right|^2 \left\{ \frac{i}{\omega_j - \omega_C} + \pi\delta\left(\omega_j - \omega_C\right) \right\} \equiv \frac{\gamma}{2} + i\Delta\omega \end{aligned} \tag{11}$$

This is the so-called Wigner-Weiskopff approximation [11]. It says there is a real contribution representing decay or dissipation of the t=0 state at a rate proportional to γ and that that decay in instantaneous or Markovian. In addition, due to interactions between the single mode radiation field and the many body reservoir, the characteristic frequency of the single mode field is shifted. This is analogous to the lamb shift [11]. If we ignore the frequency shift, substituting equation (11) into equation (9) and performing the inverse Laplace transform we obtain the following quantum Langevin equation

$$\frac{dA(t)}{dt} = -\frac{\gamma}{2}A(t) + G_A(t) \tag{12}$$

To complete the specification of our problem, we need to understand the role of the fluctuations. We do this next.

2.3. *Quantum Noise and Properties of the Reservoir*

The fluctuations of appearing in equations (7) and (12) depend only on the initial state of the many body reservoir. In order to understand the quantum Langevin equation fully, we need to specify the statistical properties of the single mode radiation field and the many body reservoir. The usual way of doing this is to define the density operator for the universe. The universe will be divided into a system + reservoir. The system or resolved degrees of freedom refer to the single mode radiation field. We assume initially that the system and reservoir are in thermodynamic equilibrium and not in contact with each other. Therefore, we have

$$\rho_{U=S+R}(0) = \rho_R(0)\rho_S(0) = \left\{Z_R^{-1}e^{-\beta_R H_R}\right\}\rho_S(0) \tag{13}$$

Quantum mechanical averages for both system and reservoir variables can be formed thusly,

$$\left\langle\left\langle O(t)\right\rangle\right\rangle = Tr_{R,S}\left(\rho_{U=S+R}(0)O(t)\right) \qquad \left\langle O(t)\right\rangle = Tr_R\left(\rho_R(0)O(t)\right) \tag{14}$$

The reservoir degrees of freedom will assume to be initially taken from a Planckian or blackbody distribution

$$\left\langle b_j^+(0)b_k(0)\right\rangle = \delta_{jk}\Big/\left(e^{\beta_R\hbar\omega_j}-1\right) \qquad \left\langle b_j^+(0)\right\rangle = \left\langle b_j(0)\right\rangle = 0 \tag{15}$$

Using this information, the properties of the noise operator are easily derived.

$$\left\langle G_A(t)\right\rangle = 0 \text{ and } \left\langle G_A^+(s)G_A(t)\right\rangle = K(s-t) \tag{16}$$

or, in the Markov limit,

$$\left\langle G_A^+(s)G_A(t)\right\rangle = \frac{\gamma\delta(s-t)}{\left(e^{\beta\hbar\omega_C}-1\right)} \tag{17}$$

Therefore, the quantum Langevin equation (7) or (12) is now exactly specified. For example, using the quantum statistical properties of $G_A(t)$, the evolution of the quantum mechanical averages of $\mathrm{A(t), A^+(t)}$ and $\mathrm{A^+(t)A(t)}$ can be computed. We obtain,

$$\frac{d\langle A(t)\rangle}{dt} = -\gamma\langle A(t)\rangle \tag{18}$$

$$\frac{d\langle A^+(t)A(t)\rangle}{dt} = -\gamma\langle A^+(t)A(t)\rangle + \frac{\gamma}{\left(e^{\beta\hbar\omega_C} - 1\right)} \tag{19}$$

Therefore, the quantum expectation value of the annihilation and creation operators goes to zero as $t \to \infty$. Interestingly, the fluctuations or single mode radiation field number approach a blackbody at the temperature of the reservoir as $t \to \infty$.

3. The Stochastic Quantum Radiative Transfer Equation

3.1. *Micro-physical Description of a Fully Ionized Plasma*

Consider a non-relativistic fully ionized plasma whose degrees of freedom consist of free electrons, photons, interactions, and a background of ions with charge Ze. The total Hamiltonian can be written as the following sum,

$$H = H_{photons} + H_{electrons} + H_{Coulomb} + H^1_{photon-electron} + H^2_{photon-electron} \tag{20}$$

In the number representation, the various contributions to the total Hamiltonian can be written as the usual product of annihilation and creation operators.

$$H_{photon} = \sum_k \sum_\lambda \hbar\omega_k b^+_{k,\lambda}(t) b_{k,\lambda}(t) \quad H_{electron} = \sum_k \sum_\lambda E_k a^+_k(t) a_k(t) \tag{21}$$

$$H_{Coulomb} = \sum_k \sum_{k'} \tilde{V}(k-q) a^+_k(t) a_{k'}(t) + \sum_k \sum_{k'} \sum_q \tilde{U}(q) a^+_{k+q}(t) a^+_{k'-q}(t) a_{k'}(t) a_k(t) \tag{22}$$

The potential V represents the coulomb interaction between free electrons and a background sea of ions of charge Ze. The Coulomb potential U represents the electron-electron Coulomb interaction. The photon-electron interaction $H^1_{photon-electron}$ is responsible for the bare "first-order" emission and absorption processes or Bremsstrahlung and inverse Bremsstrahlung. It is given by,

$$H^1_{photon-electron} = \frac{e\hbar}{mc}\sum_k\sum_{k'}\sum_\lambda \alpha_k\left(k'\bullet\varepsilon_{k,\lambda}\right)\left[a^+_{k+k'}(t)a_{k'}(t)b_{k,\lambda}(t)+a^+_{k'}(t)a_{k+k'}(t)b^+_{k,\lambda}(t)\right] \tag{23}$$

The constant $\alpha_k = \sqrt{2\pi\hbar c^2 / V\omega_k}$ where V is the spatial volume. The quantity $\varepsilon_{k,\lambda}$ is the photon polarization vector associated with wave number k and polarization λ . There also exists another contribution to the Hamiltonian coming from bare "second order" processes involving photons and electrons. This is the lowest level contribution to scattering. Finally, since the electrons are fermions, they will obey anti-commutation relations and the photons being bosons obey commutations relations.

$$\begin{aligned}&\left\{a^+_k(t),a_{k'}(t)\right\}=\delta_{kk'}\\&\left[b_{k,\lambda}(t),b^+_{k',\lambda'}(t)\right]=\delta_{kk'}\delta_{\lambda\lambda'}\\&k\bullet\varepsilon_{k,\lambda}=0\ \text{(Coulomb Gauge)}\end{aligned} \tag{24}$$

Due to the gauge invariance of QED, we are free to choose an convenient gauge to work in. For this paper, we have chosen the Coulomb gauge.

3.2. *The Macro and Micro Variable Equations of Motion*

Using the Heisenberg equations of motion equation (2) and the commutation relations (24), we can easily obtain

$$\frac{db_{q,\gamma}(t)}{dt} = -i\omega_q b_{q,\gamma}(t)+\left(\frac{ie\theta}{mc}\right)\sum_k\left(k\bullet\varepsilon_{q,\gamma}\right)\Omega_{k,k+q}(t) \tag{25}$$

$$\frac{da_k(t)}{dt} = \left(\frac{-iE_k}{\hbar}\right)b_k(t) - \left(\frac{i\theta}{\hbar}\right)\sum_{k'} V(k-k')a_{k'}(t)$$
$$-\left(\frac{i\theta}{\hbar}\right)\sum_{k'}\sum_{q} U\left(k-k'\right)\Omega_{k'+q-k,q}(t)a_{k'}(t) \tag{26}$$
$$+\left(\frac{ie\theta}{mc}\right)\sum_{k'}\sum_{\lambda}\alpha_{k'}(k \bullet \varepsilon_{k',\lambda})\left[a^{+}_{k-k'}(t)b_{k',\lambda}(t) + b^{+}_{k',\lambda}(t)a_{k+k'}(t)\right]$$

Where we have introduced a perturbation parameter θ , which is associated with all types of photon electron interactions. In addition, we have introduced a generalized electron number operator given by $\Omega_{k,k+q} = a^{+}_{k}(t)a_{k+q}(t)$.

Following the analysis of section 2, we first identify our resolved and unresolved modes. The purpose of this work is to arrive at a form for the photon transport equation in the presence of a plasma which is not necessarily in LTE. Therefore, we identify the resolved modes with the photon degrees of freedom and the reservoir or unresolved modes with the electron degrees of freedom. Unfortunately, unlike our simple example in section 2, the equation of motion for the electrons can not be written in closed form. The origin of this difficulty is the nature of the interaction equation (23) which is no longer bi-linear. Instead, we use a perturbative method due to Lindenburg and West [12]. Using the perturbation parameter as a guide, the solution to equation (26), to first order in θ, is given by

$$a_k(t) = e^{\frac{-iE_k t}{\hbar}}a_k(0) - \left(\frac{i\theta}{\hbar}\right)\sum_{k'} V(k-k')\int_0^t d\tau e^{\frac{-iE_k(t-\tau)}{\hbar}}a_{k'}(\tau)$$
$$-\left(\frac{i\theta}{\hbar}\right)\sum_{k'}\sum_{q} U\left(k-k'\right)\int_0^t d\tau e^{\frac{-iE_k(t-\tau)}{\hbar}}\Omega_{k'+q-k,q}(\tau)a_{k'}(\tau) \tag{27}$$
$$+\left(\frac{ie\theta}{mc}\right)\sum_{k'}\sum_{\lambda}\alpha_{k'}(k \bullet \varepsilon_{k',\lambda})\int_0^t d\tau e^{\frac{-iE_k(t-\tau)}{\hbar}}\left[a^{+}_{k-k'}(\tau)b_{k',\lambda}(\tau) + b^{+}_{k',\lambda}(\tau)a_{k+k'}(\tau)\right]$$

Since the equation of motion for the photon degrees of freedom involve the quantity $\Omega_{k,k+q}$, we can use equation (27) to construct the generalized electron number operator. The expression is somewhat lengthy; however, it can be decomposed into two types of contributions. We write

$$\Omega_{k,k+q}(t) = T_{k,k+q}(t) + S_{k,k+q}(t) \tag{28}$$

The first type of contribution depends only on the initial state of the electron plasma. As seen is section 2, it is the term that we will see generates the quantum noise coming from the electron reservoir. The second type of term depends not only on the initial state of the electron plasma, but it also depends on the photon annihilation and creation operators. As we will see, it is responsible for generating the quantum damping or absorption in the photon equation of motion. Explicitly, we write,

$$\begin{aligned}
&T_{k,k+q}(t) \approx e^{\frac{i(E_k - E_{k+q})t}{\hbar}} \Omega_{k,k+q}(0) + \text{terms of order } \theta \\
&S_{k,k+q}(t) \approx \\
&-\left(\frac{ie\theta}{mc}\right)\sum_{k'}\sum_{\lambda}\alpha_{k'}(k \bullet \varepsilon_{k',\lambda})\int_0^t d\tau e^{\frac{iE_{k+k'}\tau}{\hbar}} e^{\frac{-iE_k(t-\tau)}{\hbar}} b_{k',\lambda}(\tau)\Omega_{k+k',k+q}(0) \\
&+\left(\frac{ie\theta}{mc}\right)\sum_{k'}\sum_{\lambda}\alpha_{k'}([k+q]\bullet \varepsilon_{k',\lambda})\int_0^t d\tau e^{\frac{iE_{k+k'}\tau}{\hbar}} e^{\frac{-iE_k(t-\tau)}{\hbar}} \Omega_{k,k+q-k'}(0) b^{+}_{k',\lambda}(\tau) \\
&+ \text{terms of order } \theta^2
\end{aligned} \tag{29}$$

We have kept terms in the quantity T up to first order in the expansion parameter and kept terms in the quantity S to second order in the expansion parameter. The reason for this is that terms of order θ^N that contribute to the noise, contribute to order θ^{2N} to the dissipation or absorption through the FDR. We see this in our simple example in section 2. The quantum noise $G_A \approx |\kappa_j|$ and the dissipation kernel $K \approx |\kappa_j|^2$ and both are related by a FDR (equations (16) and (17)).

3.3. *The Photon Quantum Langevin Equation*

We are now in a position to construct a Langevin equation for the photon degrees of freedom. Defining a new variable $b_{k,\lambda}(t) = e^{-i\omega_k t} B_{k,\lambda}(t)$, we obtain

$$\frac{dB_{q,\gamma}(t)}{dt} = e^{i\omega_q t}\left(\frac{ie\theta}{mc}\right)\alpha_q \sum_k \left(k \bullet \varepsilon_{q,\gamma}\right)\left[T_{k+q}(t) + S_{k+q}(t)\right] \tag{30}$$

Following our arguments about retaining terms of one less order in T versus S, we can write

$$\begin{aligned} T_{k,k+q} &\propto 1+\theta+\theta^2+\ldots \\ S_{k,k+q} &\propto \theta+\theta^2+\ldots \end{aligned} \tag{31}$$

At this point we are almost done. There are two remaining complications. One is what to do with products of operators of the form $\Omega_{k,q}(0)b_{p,\lambda}(t)$? These terms probably contribute to an interesting phenomenon called multiplicative noise [9]. This needs to be investigated further. However, we will make a simplification and make a "mean-field" approximation and write

$$\Omega_{k,q}(0) \Rightarrow n_k(\beta)\delta_{k,q} + O(\theta) \tag{32}$$

Second, we will ignore multi-photon processes coming from products of operators like $b_{k,\lambda}(t)b_{q,\gamma}(\tau)$. This also needs to be investigated further.

Using equation (32), we obtain our quantum Langevin equation for the reduced photon degrees of freedom.

$$\begin{aligned} &\frac{dB_{q,\gamma}(t)}{dt} \approx \\ &\left(\frac{e\theta}{mc}\right)^2 \alpha_q^2 \sum_k \sum_\lambda \left(k \bullet \varepsilon_{q,\lambda}\right)\left(k \bullet \varepsilon_{q,\gamma}\right)\left[n_{k+q}(\beta) - n_k(\beta)\right]\int_0^t d\tau e^{-i\Delta(k,q)(t-\tau)} B_{q,\gamma}(\tau) \\ &+ G_{q,\gamma}(t) \end{aligned} \quad (33).$$

$$G_{q,\gamma}(t) = \left(\frac{ie\theta}{mc}\right)\alpha_q \sum_k \left(k \bullet \varepsilon_{q,\gamma}\right) e^{-i\Delta(k,q)t}\Omega_{k,k+q}(0) \tag{34}$$

This equation has both damping or absorption terms and a quantum noise source similar to what we saw in our simple model. However, equation (33) has some interesting properties. One of these is the fact that something like dielectric screening has appeared in the absorption contribution. Our procedure has been able to include collective effects. That is, the appearance of terms like $\left[n_{k+q}(\beta) - n_k(\beta)\right]$ appear in the dielectric function of an electron plasma.

We will see in section 3.6 that in fact the absorption contribution in equation (33) is just the dielectric function of an electron plasma undergoing a photon mediated effective interaction.

3.4. *The Photon Quantum Langevin Equation in the Markov Limit*

We now look in the Markov limit where the characteristic relaxation time for the electron plasma is much shorter than the characteristic time of evolution for the photons. This occurs for example in dense plasmas or plasmas with large heat capacities where regardless of the photon effects on the plasma, the average properties plasma do not change but rather fluctuate around some mean value. Staring with our Langevin equation, we have

$$\begin{aligned}&\frac{dB_{q,\gamma}(t)}{dt}\approx\\&\left(\frac{e\theta}{mc}\right)^2\alpha_q^2\sum_k\sum_\lambda\left(k\bullet\varepsilon_{q,\lambda}\right)\left(k\bullet\varepsilon_{q,\gamma}\right)\left[n_{k+q}(\beta)-n_k(\beta)\right]\int_0^t d\tau e^{-i\Delta(k,q)(t-\tau)}B_{q,\gamma}(\tau)\\&+G_{q,\gamma}(t)\end{aligned} \tag{35}$$

Performing the Laplace transform, we obtain,

$$\tilde{B}_{q,\gamma}(s)\propto\frac{1}{s+\sum_k\sum_\lambda\left(k\bullet\varepsilon_{q,\gamma}\right)\left(k\bullet\varepsilon_{q,\lambda}\right)\frac{\left[n_k(\beta)-n_{k+q}(\beta)\right]}{s+i\Delta(k,q,\omega)}} \tag{36}$$

Where

$$\Delta(k,q,\omega)=\frac{\left(E_{k+q}-E_k\right)}{\hbar}-\omega_q$$

We now make the Wigner-Weiskopff approximation and assume the interaction between the photons and plasma is weak. Performing a slight shift of the pole, we obtain

$$\Gamma(s) - s \Rightarrow \lim_{s \to 0} \Gamma(s) - s =$$

$$\sum_k \sum_\lambda \left(k \bullet \varepsilon_{q,\gamma}\right)\left(k \bullet \varepsilon_{q,\lambda}\right)\left[n_k(\beta) - n_{k+q}(\beta)\right]\left[\frac{-1}{\Delta(k,q,\omega)} - i\pi\delta\left[\Delta(k,q,\omega)\right]\right] \quad (37)$$

$$\equiv \frac{1}{2}\Gamma_\gamma(q,\omega) + i\Phi_\gamma(q,\omega)$$

That is, there are two contributions one real and one imaginary. The real contribution is the absorption and the imaginary contribution is the frequency shift in the radiation field due to the plasma photon interaction. Ignoring the frequency shift as being small, we have the absorption piece,

$$\Gamma_\gamma(q,\omega) =$$

$$\pi\left(\frac{e\theta}{mc}\right)^2 \alpha_q^2 \sum_k \sum_\lambda \left(k \bullet \varepsilon_{q,\gamma}\right)\left(k \bullet \varepsilon_{q,\lambda}\right)\left[n_k(\beta) - n_{k+q}(\beta)\right]\delta\left[\Delta(k,q,\omega)\right] \quad (37)$$

We will look at some of its properties in section 3.6.

Our goal of course is to construct an equation for photon number since that is what is directly measured and that is the quantity associated with the specific intensity. Using equations (35) and (37), in the Markov limit we obtain.

$$\frac{dN_{q,\gamma}(t)}{dt} \approx -\Gamma_0(q,\gamma)N_{q,\gamma}(t) + G_{q,\gamma}^+(t)B_{q,\gamma}(t) + B_{q,\gamma}^+(t)G_{q,\gamma}(t) \quad (38)$$

This is a quantum Langevin equation for photon number. Like our simple model there is the absorption term. However, the nose term is much more complicated. We will investigate its properties in the next section.

Equation (38) forms the starting point for constructing a radiative transfer equation. We won't explicitly do it here as it is done in Graziani [7]. However, the basic steps involve first defining a generalized photon number operator

$$N_{qk,\gamma\lambda}(t) = B_{q,\gamma}^+(t)B_{k,\lambda}(t) \quad (39)$$

The connection to the specific intensity is via the Klimontovich operator.

$$I_{\gamma\lambda}(\vec{x},\vec{q},t) = \frac{\hbar\omega_q^3}{2\pi^2 c^2}\sum_k N_{qk,\gamma\lambda}(t)\left(\varepsilon_{q,\gamma} \bullet \varepsilon_{k,\lambda}\right)e^{-i\left(\vec{q}-\vec{k}\right)\bullet\vec{x}}e^{i\left(\omega_q-\omega_k\right)t} \quad (40)$$

This quantity is a specific intensity operator. It expectation value is the quantity referred to in the introduction. The time evolution equation of $I_{\gamma\lambda}(\vec{x},\vec{q},t)$, which is a quantum generalization of the radiative transfer equation discussed earlier, is obtained by differentiating equation (40) with respect to time and using a generalization of equation (38) to remove time derivatives on the right hand side of the equation.. The result is a stochastic quantum radiative transfer equation. Since the equation is in operator form, moments of $I_{\gamma\lambda}(\vec{x},\vec{q},t)$ can be calculated to compute quantum fluctuations in the photon intensity.

3.5. *Quantum Noise, Kirchoff's Law, and the FDR*

A complete specification of the quantum Langevin equation for photon number requires understanding the properties of the noise operator. In order to do this, we need to specify the initial state of the electron plasma. We assume that the plasma is initially in thermal equilibrium at a temperature T and chemical potential μ. Since the electrons are fermions, we have

$$\left\langle \Omega_{k,q}(0)\right\rangle \equiv \left\langle a_k^+(0)a_q(0)\right\rangle = \delta_{kq}n_k(\beta) \equiv \delta_{kq}\Big/\left(e^{\beta(E_k-\mu)}+1\right) \quad (41)$$

The photons will also initially be assumed to be in thermal equilibrium but at some temperature different than the electrons. Since the photons are bosons, we have

$$\begin{aligned}\left\langle\left\langle N_{q,\gamma,k,\lambda}(t)\right\rangle\right\rangle &= \left\langle\left\langle b_{q,\gamma}^+(0)b_{k,\lambda}(0)\right\rangle\right\rangle \\ &= \delta_{jk}\delta_{\gamma\lambda}J_q(\beta_P) \equiv \delta_{jk}\delta_{\gamma\lambda}\Big/\left(e^{\beta_P\hbar\omega_j}-1\right)\end{aligned} \quad (42)$$

Using equation (34), we have

$$\left\langle G_{q,\gamma}^{+}(s)G_{q,\gamma}(t)\right\rangle =$$
$$\left(\frac{e\theta}{mc}\right)^{2}\alpha_q^2\sum_k\sum_k\left(k\bullet\varepsilon_{q,\gamma}\right)\left(k'\bullet\varepsilon_{q,\gamma}\right)e^{-i\left[\Delta(k',q,\omega)-\Delta(k,q,\omega)\right]t}\left\langle\Omega_{k+q,k}(0)\Omega_{k',k'+q}(0)\right\rangle \tag{43}$$

Products of zero time annihilation and creation operators can be easily computed using the canonical commutation relations equation (24). We have

$$\left\langle G_{q,\gamma}^{+}(s)G_{q,\gamma}(t)\right\rangle =$$
$$\left(\frac{e\theta}{mc}\right)^{2}\alpha_q^2\sum_k\left(k\bullet\varepsilon_{q,\gamma}\right)^2 e^{i\Delta(k,q,\omega)(s-t)}n_{k+q}(\beta)\left[1-n_k(\beta)\right] \tag{44}$$
$$\left\langle G_{q,\gamma}^{+}(s)G_{q,\gamma}(t)\right\rangle = 2D_\gamma(q,\omega_q)\delta(s-t)$$

Where the diffusion coefficient $D_\gamma(q,\omega_q)$ is given by

$$2D_\gamma(q,\omega_q)$$
$$=\left(\frac{e\theta}{mc}\right)^{2}\alpha_q^2\sum_k\left(k\bullet\varepsilon_{q,\gamma}\right)^2 n_{k+q}(\beta)\left[1-n_k(\beta)\right]2\pi\delta\left[\Delta(k,q,\omega_q)\right] \tag{45}$$

We now compute the noise term appearing in equation (38) and we will show that it is simply related to the above diffusion coefficient. We need to evaluate

$$\left\langle G_{q,\gamma}^{+}(t)B_{q,\gamma}(t)+B_{q,\gamma}^{+}(t)G_{q,\gamma}(t)\right\rangle = ? \tag{46}$$

The procedure is straightforward. We first use the definition of $G_{q,\gamma}(t)$ and write

$$G_{q,\gamma}(t) = \left(\frac{ie\theta}{mc}\right)\alpha_q \sum_k \left(k \bullet \varepsilon_{q,\gamma}\right) e^{-i\Delta(k,q,\omega_q)t} \Omega_{k,k+q}(0) \qquad (47)$$

We then write down the solution of equation (33) when the Markov approximation is invoked. The solution is

$$B_{q,\gamma}(t) = e^{-\frac{\Gamma_\gamma(q,\omega)t}{2}} B_{q,\gamma}(0) + \int_0^t d\tau e^{-\frac{\Gamma_\gamma(q,\omega)(t-\tau)}{2}} G_{q,\tau}(\tau) \qquad (48)$$

Substituting equations (48) and (47) into equation (46) and using equation (44), we have

$$\left\langle G^+_{q,\gamma}(t) B_{q,\gamma}(t) + B^+_{q,\gamma}(t) G_{q,\gamma}(t) \right\rangle = 2D_\gamma(q,\omega_q) \qquad (49)$$

Therefore, we see that the noise term $\left\langle G^+_{q,\gamma}(t) B_{q,\gamma}(t) + B^+_{q,\gamma}(t) G_{q,\gamma}(t) \right\rangle$ is not zero centered. We now look at the diffusion coefficient more carefully.

We have

$$2D_\gamma(q,\omega_q)$$
$$= \left(\frac{e\theta}{mc}\right)^2 \alpha_q^2 \sum_k \left(k \bullet \varepsilon_{q,\gamma}\right)^2 n_{k+q}(\beta)\left[1 - n_k(\beta)\right] 2\pi\delta\left[\Delta(k,q,\omega_q)\right] \qquad (50)$$

We replace the summation over momentum by an integration and define an orthonormal coordinate system in momentum space whose basis vectors are, $q/|q|, \varepsilon_{q,1}$ and $\varepsilon_{q,2}$. Transforming to a new variable $u = \beta E_k$ and choosing q to lie along the z axis, we obtain in spherical coordinates in momentum space,

$$D_\gamma(q,\omega_q)$$
$$= 2\pi\left(\frac{e}{mc}\right)^2 \alpha_q^2 \frac{V}{8\pi^2} \frac{m}{\hbar q}\left(\frac{m}{\beta\hbar^2}\right)^2 e^{-\mu\beta} \int_{u_{\min}}^{\infty} du \frac{1}{\left(ae^u + 1\right)\left(be^u + 1\right)} \qquad (51)$$

Where

$$u_{\min} = \beta \frac{\hbar^2 q^2}{8m}\left[\left(\frac{\hbar\omega_q}{\hbar^2 q^2/2m}\right) - 1\right] \text{ and } a = e^{-\mu\beta} \quad \text{and} \quad \mathrm{b} = e^{-\mu\beta + \beta\hbar\omega_q}$$

The constraint on the range of the integration comes from the delta-function appearing in equation (50) In addition, although we have kept the polarization index on the diffusion coefficient appearing in equation (51), equation (51) means that

$$D_{\gamma=1}(q,\omega_q) = D_{\gamma=2}(q,\omega_q)$$

We now convert the expression for $\Gamma_\gamma(q,\omega_q)$ to an integral using the same steps. We obtain

$$\Gamma_\gamma(q,\omega_q) = 2\pi\left(\frac{e}{mc}\right)^2 \alpha_q^2 \frac{V}{8\pi^2}\frac{m}{\hbar q}\left(\frac{m}{\beta\hbar^2}\right)^2 \int_{u_{\min}}^{\infty} du \frac{[b-a][u-u_{\min}]e^u}{(ae^u+1)(be^u+1)} \tag{52}$$

Using the simple identity

$$\frac{e^{-\mu\beta}}{(b-a)} = \frac{1}{e^{\beta\hbar\omega_q}-1}$$

We obtain the relationship

$$2D_\gamma(q,\omega_q) = \frac{\Gamma_\gamma(q,\omega_q)}{\left(e^{\beta\hbar\omega_q}-1\right)} \tag{52}$$

Again, the above relationship holds for all polarization states. Note that the Planck function which appears in equation (52) depends on the plasma temperature, not the radiation temperature. This expression does two things. One it establishes a relationship between the absorptivity or dissipation and the fluctuations; the fluctuation dissipation relation. That is

$$\left\langle G^{+}_{q,\gamma}(s)G_{q,\gamma}(t)\right\rangle = 2D_{\gamma}(q,\omega_q)\delta(s-t) = \frac{\Gamma_{\gamma}(q,\omega_q)}{\left(e^{\beta\hbar\omega_q}-1\right)}\delta(s-t) \quad (53)$$

Second, through equation (52), we have a re-statement of Kirchoff's Law. That is,

$$\left\langle G^{+}_{q,\gamma}(t)B_{q,\gamma}(t) + B^{+}_{q,\gamma}(t)G_{q,\gamma}(t)\right\rangle = 2D_{\gamma}(q,\omega_q) = \frac{\Gamma_{\gamma}(q,\omega_q)}{\left(e^{\beta\hbar\omega_q}-1\right)} \quad (54)$$

That is the average of the quantum noise operator appearing in the photon number Langevin equation (equation (38)) acts like an emissivity given by a product of a blackbody at the plasma temperature multiplied by an absorptivity! This can be seen more clearly if we define a quantum noise operator that is zero centered by writing

$$F_{q,\gamma}(t) = G^{+}_{q,\gamma}(t)B_{q,\gamma}(t) + B^{+}_{q,\gamma}(t)G_{q,\gamma}(t) - \frac{\Gamma_{\gamma}(q,\omega_q)}{\left(e^{\beta\hbar\omega_q}-1\right)} \quad (55)$$

Therefore, the quantum Langevin equation for photon number is given by

$$\begin{gathered}\frac{dN_{q,\gamma}(t)}{dt} \approx -\Gamma_0(q,\gamma)N_{q,\gamma}(t) + \frac{\Gamma_0(q,\gamma)}{\left(e^{\beta\hbar\omega_q}-1\right)} + F_{q,\gamma}(t) \\ \left\langle F_{q,\gamma}(t)\right\rangle = 0\end{gathered} \quad (56)$$

Therefore, the new quantum noise operator describes the NLTE emissivity as quantum fluctuations around Kirchoff's law. In addition, equation (56) describes the relaxation of an initially out of equilibrium photon gas towards an equilibrium blackbody with a temperature characteristic of the plasma.

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3.6. *The Absorption Coefficient*

In this section, we derive a closed form for the absorptivity. To re-iterate, we have

$$\Gamma_{\gamma}(q,\omega_q) =$$
$$\pi\left(\frac{e\theta}{mc}\right)^2 \alpha_q^2 \sum_k \sum_\lambda \left(k \bullet \varepsilon_{q,\gamma}\right)\left(k \bullet \varepsilon_{q,\lambda}\right)\left[n_k(\beta) - n_{k+q}(\beta)\right]\delta\left[\Delta(k,q,\omega_q)\right] \quad (57)$$

Due to the presence of the delta-function in the summation over momentum, not all values of k are allowed. They must obey the following constraint,

$$-1 \le \left[\frac{q}{2k} - \frac{m\omega_q}{\hbar q k}\right] \le 1 \quad (58)$$

We transform to the variable of integration $u = \beta E_k$,

$$\Gamma_{\gamma}(q,\omega_q)$$
$$= 2\pi\left(\frac{e}{mc}\right)^2 \alpha_q^2 \frac{V}{8\pi^2}\frac{m}{\hbar q}\left(\frac{m}{\beta\hbar^2}\right)^2 e^{-\mu\beta} \int_{u_{\min}}^{\infty} du \frac{\left[u - u_{\min}\right]e^u}{\left(ae^u + 1\right)\left(ae^u + 1\right)} \quad (59)$$

$$u_{\min} = \beta\frac{\hbar^2 q^2}{8m}\left[\left(\frac{\hbar\omega_q}{\hbar^2 q^2 / 2m}\right) - 1\right] \quad a = e^{-\mu\beta} \quad \text{and} \quad \mathrm{b} = e^{-\mu\beta + \beta\hbar\omega_q} \quad (60)$$

Transform variables again to $z = e^{-u}$ yields,

$$\Gamma_{\gamma}(q,\omega_q)$$
$$= -2\pi\left(\frac{e}{mc}\right)^2 \alpha_q^2 \frac{V}{8\pi^2}\frac{m}{\hbar q}\left(\frac{m}{\beta\hbar^2}\right)^2 e^{-\mu\beta} \int_0^{e^{-u_{\min}}} dz \frac{\left[\ln(z) + u_{\min}\right]}{\left[a + z\right]\left[b + z\right]} \quad (61)$$

Use method of partial fractions and write

$$\frac{1}{[a+z][b+z]} = \frac{1}{(b-a)}\left(\frac{1}{a+z} - \frac{1}{b+z}\right) \tag{62}$$

Use the following identities,

$$\int_0^x \frac{dz}{a+z} = \ln(1+\frac{x}{a}) \quad \text{and} \quad \int_0^x \frac{dz\ln(z)}{a+z} = Li_2\left(-\frac{x}{a}\right) + \ln(x)\ln(1+\frac{x}{a})$$

We obtain

$$\Gamma_\gamma(q,\omega_q) = \frac{e^2 m}{2\omega_q q\beta^2\hbar^4}\left[Li_2\left(-\frac{e^{-u_{\min}}}{b}\right) - Li_2\left(-\frac{e^{-u_{\min}}}{a}\right)\right]\frac{1}{\left(e^{\beta\hbar\omega_q}-1\right)} \tag{63}$$

In Figure 1, the absorption coefficient (in units of 1/sec) is plotted as a function of photon energy $\hbar\omega_q$ for a plasma with a number density of $n = 10^{18}/cc$. Figures A, B, C and D have $E_q = \hbar^2 q^2/2m = 1eV, 100\,\text{eV}, 1\,\text{keV}, \text{and}\, 10\text{keV}$ and a plasma temperature of 10 eV. A multiplier has been placed on curves C and D of 10^2 and 10^{12} respectively. Clearly, absorptivity decreases strongly as a function of q. In addition, for a fixed q, there is a power law fall off as a function of $\hbar\omega_q$ for all curves. However, for $1/\beta > E_q$, there is a value of $\hbar\omega_q$ for which $\Gamma_\gamma(q,\omega_q)$ falls precipitously. For temperatures $1/\beta < E_q$ the curves are mono-tonic, exhibiting a slight peak for $E_q = 1keV$ and a pronounced peak for $E_q = 10keV$

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6. Figure Caption

Figure 1. Absorption coefficient (in units of 1/sec) as a function of photon energy $\hbar\omega_q$ for a plasma with a number density of $n = 10^{18}\,/\,cc$. Figures A, B, C and D have $E_q = \hbar^2 q^2 / 2m = 1eV, 100\,\mathrm{eV}, 1\,\mathrm{keV}, \text{and}\,10\mathrm{keV}$ and a plasma temperature of 10 eV. A multiplier has been placed on curves C and D of 10^2 and 10^{12} respectively.